Frax

SOLVING THE FRACTIONS PROBLEM

Research Summary & Frax Classroom Results

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Highlights

1

Competency with fractions is a strong predictor of long-term success in math.
Numerous studies have shown that fractions knowledge is integral to and predictive of success in algebra.

2

Fractions are when math stops making sense to a lot of students.
A lack of understanding of fractions as numbers leads students to treat numerators and denominators as separate entities, causing errors in mathematical thinking like believing 10/4 is greater than 3 because 10 and 4 are both greater than 3.

3

The most important representations when teaching fractions are length models and number lines.
Instructional methods that show the strongest evidence of success use a measurement conceptualization of fractions, with number lines as the primary representation, instead of the more traditional part-whole interpretation.

4

ExploreLearning Frax applies research-based methods to a gamified learning experience, leading to greater positive classroom results in a quarter of the time of other programs.

5

Best of all, students and teachers love Frax!

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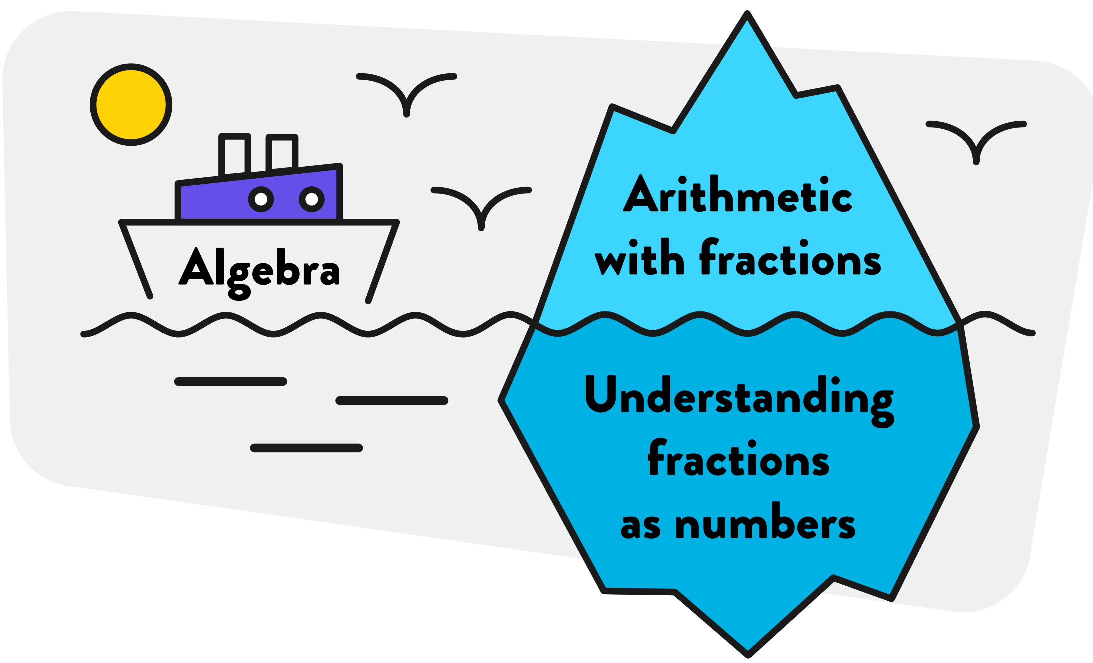
Foundations for Success:
The Final Report of the National Mathematics Advisory Panel (2008)



Executive Summary

Algebra is the gateway to higher study in mathematics. Rising failure rates, however, are shutting this gate for more and more students across the country – significantly narrowing college options and cutting off lucrative, growing STEM career fields.

Concerns over these declines in U.S. math performance led to the formation of the National Math Advisory Panel (NMAP) in 2006. Based on a two-year investigation that included a review of over 16,000 research publications and policy documents as well as a nationally representative survey of algebra teachers, the NMAP's Final Report (2008) identified key gaps in rising middle schoolers' readiness for algebra – with **fractions proficiency the critical 'missing pillar.'**

Subsequent studies following large samples of students through middle and high school gave further confirmation of this finding. **Competency with fractions** measured at the end of elementary school **was identified as a strong predictor of long-term success in mathematics**, even after controlling for variables that typically affect performance (including competency with whole numbers, general IQ, and family income & educational level).

Why is this critical pillar for advanced study missing for so many students? One factor can be traced to the ubiquitous 'pizza' and 'pie' fraction models that have long been a rite of passage in elementary school. Instruction has often emphasized the part-whole conceptualization of a fraction that these models focus on. This approach often instills **no real understanding of fractions as numbers**, a deficit clearly seen in the dismal performance over the past 50 years in National Assessment of Educational Progress questions on fraction fundamentals. This lack of conceptual understanding then creates cascading problems for students as they move into operations with fractions. They are left memorizing abstract rules – seemingly devoid of any real meaning – resulting in an ever-growing mental pile of disconnected procedures. Math stops making sense, and studies show negative attitudes and anxiety associated with fractions taking hold, persisting into adulthood.

A surge in research on effective fractions instruction in the years since the NMAP's report has led to a deeper understanding of what genuine mastery entails. To fully integrate fractions into an expanded concept of the number system, students must:

- engage in a progressive series of partitive and iterative activities with models of fractional lengths, including extensive work constructing fractions greater than 1;
- learn to place fractions with a broad range of denominators on a number line, understanding how the number system has now expanded beyond whole numbers to become infinitely dense (an infinite number of values existing between any two points on the number line);
- show proficiency with equivalence and magnitude comparisons between fractions, whole numbers, and mixed numbers, and justify their reasoning using models or representations;
- estimate fractions' positions on number lines, to demonstrate a fully-developed number sense with fractions and ability to think of them as numbers in their own right;
- be able to explain rules for operations with fractions using models or representations.

It is a tall order to expect all students to be able to achieve this level of mastery in today's classrooms, given the difficulty so many have historically experienced – but in 2021, the Department of Education's Institute of Educational Sciences summarized a strong body of evidence that had emerged across successful experimental interventions over the past two decades. These interventions targeted varied aspects of students' conceptual understanding of fractions and achieved noteworthy results through systematic instruction anchored around a **measurement interpretation of fractions**, with the number line as its main representation.

There can be considerable obstacles to overcome when implementing these research-validated methods in traditional classroom settings, however. For example, to learn from fraction models and number lines, studies have shown that students must work with **accurately drawn representations** and receive **immediate feedback**, particularly with activities involving *estimation*. To place operations with fractions on a solid conceptual foundation, numerical procedures need to be scaffolded with models and other semi-concrete representations that dynamically reflect a student's work in progress on a solution and then progressively faded as competency is demonstrated. Finally, providing individualized progressions through fraction learning activities is essential. This ensures that students achieve all the necessary proficiency milestones in a workable timeframe for a classroom setting.

Adaptive interactive learning environments like ExploreLearning Frax can meet these challenging requirements, bringing a systematic approach to bear across the entire range of fraction fundamentals. By integrating research-validated methodologies into a single, efficient system, Frax makes a high level of mastery an attainable reality for all.

ExploreLearning Frax gives instructional leaders the means to preemptively address a principal barrier impeding so many students today – **proficiency with fractions** – and enables teachers to implement proven solutions in a manageable manner in their classrooms. With a solid foundation for higher learning, students will thrive in their math studies for the remainder of the year and beyond.

1.1 Algebra: the gatekeeper to higher study and prosperous futures

Competency with fractions is a strong predictor of long-term success in mathematics. In particular, it plays a critical role in algebra readiness.

In 2006, the National Math Advisory Panel (NMAP) was charged under a Presidential Executive Order to conduct an extensive review of mathematics curricula and teaching methods. A key driver for the panel's formation was the perception of a general loss in American competitiveness attributable to students' declining proficiency in mathematics:

> *During most of the 20th century, the United States possessed peerless mathematical prowess...*
> *But without substantial and sustained changes to its educational system, the United States will relinquish its leadership in the 21st century. (Executive Summary of the NMAP Final Report, 2006)*

Over a two-year period, the NMAP's 25 diverse members reviewed more than 16,000 studies to identify pedagogical best practices. A specific focus was placed on research around algebra, considered the gateway to advanced study of STEM subjects in high school. As the panel wrote:

> *A strong grounding in high school mathematics through Algebra II or higher correlates powerfully with access to college, graduation from college, and earning in the top quartile of income from employment. The value of such preparation promises to be even greater in the future. The National Science Board indicates that the growth of jobs in the mathematics-intensive science and engineering workforce is outpacing overall job growth by 3:1. (NMAP Final Report, p. xii)*

Algebra's importance as a gatekeeper to higher study and a prosperous future is unfortunately matched, however, by the difficulty many experience with it. Algebra courses have a high failure rate across the United States, and the subject remains a challenge even in countries that exhibit relatively strong performance in mathematics more generally (Barbieri et al., 2021). Although students may struggle with many aspects of mathematics, the especially sharp drop off in achievement that occurs around the late middle school grades (as students encounter algebra) makes it a central concern for observers of educational policy (NMAP, 2008). The Executive Order underlying the NMAP's work placed clear emphasis on the need to address the nation's poor performance here.

1.2 The hidden role of fractions in long-term success

In fulfilling their mandate, the National Math Advisory Panel (NMAP) looked at factors associated with performance within algebra courses themselves and also students' readiness for these courses from prior study during elementary school. This led to the key finding that "the most important foundational skill not presently developed appears to be proficiency with fractions" (NMAP Final Report, 2008, p.18). Students' abilities here were assessed as "severely underdeveloped" (p. xvii) relative to the level required for success in algebra. The significance of the role played by fractions in algebra readiness led the panel to conclude unequivocally:

> *The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in algebra can be expected. (p. 18)*

This call to place a bright spotlight on 'the fractions problem' was not an isolated one, as the National Council of Teachers of Mathematics (NCTM) had also recently drawn attention to it – allotting fractions a central role from grades 3 through 6 in its intentionally more compact set of Curriculum Focal Points (2006). The Common Core State Standards for Mathematics (2010) placed a similar priority on fractions mastery during the elementary grades in its move away from existing curricula that were felt to be 'a mile wide and an inch deep.'

The NMAP's investigation into algebra readiness also included a nationally representative survey of Algebra teachers themselves. Responses to this survey bolstered the panel's findings regarding the critical need to focus on fractions; the 743 Algebra I teachers involved specifically cited extremely poor preparation in "rational numbers and operations involving fractions and decimals" in their judgment of rising middle school students' weak overall readiness for algebra (p. xix). Students' limited understanding of fractions was ranked by these teachers as one of the most important deficiencies that could and should be addressed in improving readiness.

The research literature in mathematics education had arguably lagged behind this broad consensus among algebra educators in the field, as no quantitative evidence of specific relationships between *fractions proficiency* and *algebra readiness* had been published before the NMAP's national teacher survey – but researchers soon caught up with teachers' classroom observations. A pivotal study by Booth & Newton entitled *"Fractions: could they really be the gatekeeper's doorman?"* was the first to identify significant correlations between middle school students' conceptual understanding of fractions as numbers and fundamental skills associated with algebra readiness, leading them to conclude:

> *...it is knowledge of fraction magnitudes—more so than whole number magnitude—that is related to students' skill in early algebra. (Booth and Newton, 2012, p.251)*
>
> *...students' magnitude knowledge of unit fractions (i.e., those with a numerator of 1) appears particularly important. (p. 247)*

Subsequent research deepened the evidence around the role of fractions in more advanced coursework. A major study undertaken during this period investigated aspects of student achievement at the end of elementary school that were particularly important to success in high school mathematics. Using two large educational data sets from the United States and the United Kingdom, Siegler et al. (2012) analyzed mathematics achievement in grade 5 and compared it to performance by the same students ~5 years later in high school. In both the U.S. and U.K. data sets, **fractions knowledge at the end of grade 5 was identified as a strong and unique predictor of success many years later in high school mathematics**. Critically, this relationship held even after controlling for other variables that typically impact achievement in mathematics, including knowledge of whole numbers, IQ, working memory, reading comprehension, and family income and educational levels.

Recent studies (e.g., Booth et al., (2014); Cirino et al., (2019); Hurst & Cordes (2018a; 2018b); Powell et al., (2019)) continue to enhance understanding of the unique role of fractions mastery in downstream success, particularly in algebra– as Barbieri et al. (2021) summarized:

> *Taken together, these findings suggest that the relation between number knowledge and algebra learning cannot simply be explained by an expected correlation between any two mathematics performance measures. Rather,* ***there is something unique about knowledge of fractions, in particular, that sets students up for success in algebra...*** *(p. 1985, emphasis mine)*

Fractions' importance is not restricted to the pursuit of higher mathematics and STEM careers. A comprehensive study of general workplace requirements – *the Skills, Technology and Management Practices* (STAMP) survey (Handel, 2016) – revealed that while less than one-fifth require knowledge of algebra or statistics, **more than two-thirds of current U.S. jobs involve some degree of competency with fractions**.

Outside of the workplace, researchers have also begun to examine fractions in the context of everyday adult numeracy. *Numeracy* refers to the ability to understand and use numbers in an analogous manner to literacy and reading text. It plays a key role in public comprehension of issues such as those involving medical information:

> *Individuals with limited literacy skills are at a marked disadvantage in this information age. Low literacy is associated with inferior health knowledge and disease self-management skills, and worse health outcomes...* ***A basic understanding of numerical concepts is arguably as important for informed decision making as literacy.*** *(Reyna et al., 2009, p. 4, emphasis mine)*

For example, a recent study found that accuracy in making magnitude comparisons with fractions was an important factor in decision-making in health-related and other scenarios (Thompson et al., 2021). **Having a basic understanding of and comfort with fractions must be considered a fundamental component of adult numeracy in the 21st century.**

Jacob's Story

Studies show students experience numerous challenges in learning about fractions. This next section of the white paper articulates the impact these challenges can have on children. Through the case study of "Jacob", we'll follow a hypothetical student on his journey through mid-to-late elementary school. Then we'll look at the underlying research it represents.

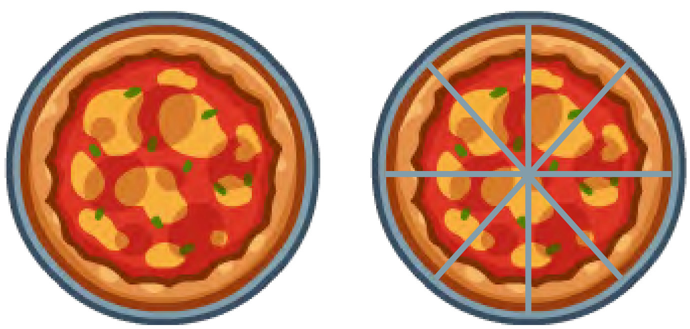
Up to now, Jacob had really liked math. Through his early elementary school work with counting (natural) numbers, he'd quickly developed confidence with the number system – everything seemed to make sense and have consistent rules (Van Hoof et al., 2017b):

- one number always followed another (3 follows 2, with no number between them);
- longer numbers always expressed greater values than shorter numbers (100 > 99);
- increasing any digit in a number always increased the magnitude (value) of that number.

Operations on numbers made sense to him too: adding and multiplying made things bigger; subtracting and dividing made them smaller. It all fit together nicely.

Then along came fractions.

His grade 3 teacher introduced them as pizza slices:



On the first fractions worksheet, he shaded pre-divided circles to match 'numerators,' and another where he divided them himself to match 'denominators.' They even did one where they shaded shapes other than circles, mostly rectangles, but also an octagon, a new shape he had recently learned about. This didn't seem too hard yet, although Jacob sometimes forgot whether the 'numerator' meant the top or bottom number – but then he noticed the top number was never larger than the bottom number on the worksheets, and after that he didn't make any mistakes – the smaller number always indicated how many to shade. It made sense, you obviously couldn't eat nine slices of a pizza cut into eighths!

After the class was done with shading shapes, the next few weeks' lessons turned to fractions in numerical form. Initially, things seemed to make sense to Jacob, like when the class had to write inequality symbols between fractions to identify which was larger:

$$\frac{2}{4} \;?\; \frac{3}{4} \qquad\qquad \frac{1}{2} \;?\; \frac{4}{6}$$

No problem with the one on the left. Obviously 3 was more than 2. But the right one seemed harder. Suddenly Jacob remembered the fraction pizza models he had worked with at the start of the unit: when the top and bottom numbers were the same, like 2/2, it had meant the whole pizza. Since the gap between 1/2 and 2/2 was only one 'slice' of pizza, and the gap between 4/6 and 6/6 was two 'slices', 1/2 must be larger than 4/6 – it was just 1 slice away from a whole pie!

He wrote "1/2 > 4/6" and quickly completed the rest of the worksheet. So far so good, he thought to himself.

At the fraction unit's end, Jacob's grade 3 class did a final brain teaser worksheet. One question asked whether 2/2 or 5/8 was closest to 1/2. Jacob confidently circled 2/2... "Not much of a brain teaser there," he thought.

In 2009, the Grade 4 National Assessment of Educational Progress included the following question:

Which fraction has a value closest to $\frac{1}{2}$ *?*

A) $\frac{5}{8}$ B) $\frac{1}{6}$ C) $\frac{2}{2}$ D) $\frac{1}{5}$

Only 25% of respondents selected the correct answer A; the most common response (41%) was C.

The last question stumped him, though – it asked for a fraction between 1/4 and 2/4. That doesn't make sense, he thought – there's no number between 1 and 2, so didn't 2/4 'follow' 1/4, with no number in between? He decided it must be a trick question and wrote "none."

Luckily, there was little time to explore these brain teasers as each year there were a slew of math topics to cover and the class quickly moved on from fractions.

Before Jacob knows it, he's in grade 5 and struggling to learn how to add fractions together. It's weird, he thinks – to add a fraction, you sometimes have to use multiplication first? His teacher cautions him that whenever he multiplies one number in a fraction, he has to remember to multiply the other one too, just like last year when he learned about something called equivalent fractions – these were fractions that meant the same amount but used different numbers. Jacob sort of understood that, but he didn't see a connection to the addition problem he was doing now.

He tried to remember what to do after you finished multiplying some of the numbers – was it add the top ones, but not the bottom ones? Or was it the other way around?

Later he learns that to divide fractions, you 'flip' them first. But he forgets which fraction to flip. Maybe both? Maybe it doesn't matter. But what does matter is that his math grades seem to be going down every year...

At the end of grade 5, Jacob sees a really strange multiple-choice question on the state test:

$$\frac{12}{13} + \frac{7}{8} \text{ is closest to}$$

A) 1 B) 2 C) 19 D) 21 E) I don't know

He's never seen a fraction with a denominator of 13 before, or had to find a number "closest to" a sum instead of just calculating the actual answer. He would love to circle "I don't know" and move on, but these days he needs every correct answer he can get just to pass math.

He starts trying to find a 'common denominator.' (That's usually the first step, right?) But 13 and 8 are really big numbers to work with, and there's no time for that! He looks at the multiple-choice answers and notices that 19 is the sum of the fractions' top numbers. "Ok, I guess that one has to be closest to the correct answer" – and thankfully that means he doesn't need to find a common denominator! Another trick question solved...

That night, his parents ask Jacob how his math test went.

He confides to them that ever since fractions showed up, math seems mainly about memorizing rules and tricks. He's still passing, but maybe he's just not a 'math person.'

His parents don't say it, but they never liked fractions when they were young, either. "It's strange, though, how different it is at work", thinks Jacob's mom. She's an interior designer and deals with fractions like ½ and ¾ all the time – she uses a tape measure pretty much every day. Jacob's dad is a chef, and lots of recipes have fractional measurements. He's never had a problem there, even when doubling a recipe – after all, 1 ½ cups is one and a half cups, right?

But for some reason, all those worksheets of fractions problems they did back in their school days had always felt like they came from some alien universe...

1.3 Where math stops making sense for many

Jacob is a hypothetical student.

However, the unfortunate reality is that students struggling like Jacob are the norm, not the exception. He struggled to understand fractions as numbers when he first learned them and that impacted him in later grades.

Almost 50 years ago, the 1978 National Assessment of Educational Progress (NAEP) for grade 8 included a multiple-choice item like the one shown below. Sadly, only 24% of students nationwide answered correctly that year – worse than random guessing, assuming exclusion of "I don't know" as a response. Given this was a grade 8 test item, this dismal performance occurred after the students had worked with fractions in a variety of capacities over the previous 5 years.

$\frac{12}{13} + \frac{7}{8}$ is closest to

A) 1 B) 2 C) 19 D) 21 E) I don't know

Performance has not substantially improved in the intervening decades. A recent study that administered the same multiple-choice question showed comparable results, with only 27% answering correctly (Lortie-Forgues et al., 2015).

These poor outcomes are not outliers restricted to the estimation of fraction sums – in examining results on any of the NAEP assessments in recent years, one readily sees the majority of students puzzled even by the simplest questions involving a fraction's magnitude. Clearly, in grasping at numerators and denominators as separate, unconnected values, **students do not understand quantities represented in fractional notation as *numbers*.**

This deficiency can be contrasted with the way in which students more readily develop 'number sense' with the natural (counting) numbers during early elementary school. Students also generally grasp the logic of operations on these numbers – with their experiences here leading many to begin noting patterns such as *multiplication makes quantities bigger, division makes them smaller*.

Then, just as students are getting a solid footing with natural numbers, rational numbers appear on the scene, in their first guise as fractions. Suddenly, commonsense relationships can appear to be overturned, and students' developing reasoning abilities with natural numbers may begin to work against them. As Van Hoof et al. (2017a, 2017b) summarize, they can no longer rely on what were seeming 'facts' to them:

- An increase in a digit doesn't always increase a number's magnitude – with natural numbers, 716, 725, and 815 are all greater than 715, but with rational numbers, 1/4 < 1/3, even though 4 > 3.
- 'Longer' numbers may or may not be greater numbers (9/2 < 10/2, but 2/9 > 2/10).
- One number no longer 'follows' another, the way 3 used to follow 2, with no other number between them – now an infinite number of values exists between any two points on the number line.

Extensive research identifies whole number bias—the applying of natural number reasoning to fractions, as in the examples above—as a common error made with magnitude comparisons that often persists into later grades, making it more difficult to build an understanding of fractions.

> *...numerous studies indeed indicate that learners make systematic and predictable errors in rational number tasks where the use of prior natural number knowledge leads to the incorrect answer...the rational number system is the first new number system learners encounter after the natural number system...**to comprehend the rational number system, learners have to construct an entirely new number concept and new procedures to handle numbers**, which are not always in accordance with their prior knowledge of (natural) numbers... (Van Hoof et al. 2017a, p.182, emphasis mine)*

The importance of integrating fractions into an evolved understanding of the number system can be considered in the context of problems such as the 1978 NAEP question described previously. Students who understand fractions as numbers would readily notice that **12/13 and 7/8 are very close to 1**. Now able to properly draw on an understanding of whole number addition, they would know that **adding two values close to 1 should result in a sum close to 2**.

$\frac{12}{13} + \frac{7}{8}$ is closest to

A) 1 B) 2 C) 19 D) 21 E) I don't know

It is also instructive to look at the most common responses to this question, which were 19 and 21. A sum of 19 is the result if one adds the numerators and disregards the denominators, with 21 the outcome if one focuses on the denominators instead. Errors like this highlight two of the challenges students face in working with fractions:

- the lack of understanding of fractions as numbers leads them to haphazardly act on numerators or denominators as entities completely divorced from one another;
- their inability to estimate a reasonable outcome for an operation on fractions precludes any realization of how gross an error sums of 19 or 21 actually are.

Thinking of fractions as two separate whole numbers also leads to significant errors like gap thinking (i.e., reasoning that 1/2 > 4/6 because the "gap" between 1 and 2 is less than that between 4 and 6). This is a way in which a lack of fraction number sense begins to inhibit subsequent learning. Without this foundation, students can only succeed with fraction arithmetic by trying to memorize a growing set of seemingly disconnected procedures that don't really make sense to them:

> *When we add or subtract fractions, we have to find a common denominator, but not when we multiply or divide. And once we get a common denominator, we add or subtract the numerators, but not the denominators, despite the fact that when we multiply, we multiply both the numerators and the denominators, and when we divide, we divide neither the numerators nor the denominators (Siebert & Gaskin, 2006, p. 394)*

The Institute of Education Sciences practice guide for fractions instruction identifies a wide range of **common misconceptions and associated errors** with operations involving fractions (Siegler et al., 2010, pp. 31-33)

Common misconception	Example of error
Believing that fractions' numerators and denominators can be treated as separate numbers	$\frac{2}{4} + \frac{5}{4} = \frac{7}{8}$
Failing to find a common denominator when adding or subtracting fractions with unlike denominators	$\frac{3}{5} - \frac{1}{2} = \frac{2}{3}$
Treating the denominator identically in both addition and multiplication problems	$\frac{2}{3} + \frac{1}{3} = \frac{3}{3}$, $\frac{2}{3} \times \frac{1}{3} = \frac{2}{3}$
Only manipulating whole numbers in computations with mixed numbers	$5\frac{3}{5} - 2\frac{1}{7} = 3$
Assuming a whole number has the same denominator as the problem's fraction	$4 - \frac{3}{8} = \frac{4}{8} - \frac{3}{8}$
Misapplication of 'invert and multiply' method for fraction division:	
Forgetting to invert the second fraction before multiplying	$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{2}{3}$
Inverting the first fraction instead of the second	$\frac{3}{4} \div \frac{2}{3} = \frac{4}{3} \times \frac{2}{3}$
Inverting both fractions	$\frac{3}{4} \div \frac{2}{3} = \frac{4}{3} \times \frac{3}{2}$

Struggling under a pile of rule on top of ungrounded rule, it is little wonder why students become confused about what to do and when to do it. Then, when they can't accurately recall a memorized method, or are faced with a problem in a form that is even slightly unfamiliar or novel to them, there is only one recourse: to grasp at answers like 19 and 21 for 12/13 + 7/8, even if they make no sense for a question involving the sum of two numbers that are both less than 1. But at least they seem to have something to do with the addition students remember from their work long ago with natural numbers – back when math made sense.

At the end of Jacob's story, even though he is doing worse, he's still passing math. However, now he thinks maybe he's just not a "math person" due to his struggle with fractions.

Another important consequence of inadequate understanding is the development of strong negative attitudes towards fractions themselves. Across a series of recent studies, Sidney et al. (2021) found that both children and adults exhibited less favorable attitudes toward any value presented as a fraction, compared to whole numbers and percentages. The participants in the studies:

- believed they were less competent with fractions;
- liked fractions less;
- thought fractions were less important and useful in daily life.

This negative view of fractions simply as a number type may even extend to elementary school teachers themselves, who report higher anxiety when teaching fractions compared to natural numbers (Thompson et al., 2021). Parents and educators who are anxious around fractions risk passing this anxiety on, perpetuating the longstanding fractions problem in each successive generation of students to come.

A better approach is clearly needed. The next section of this white paper summarizes exciting recent research on effective pedagogical methods for fraction mastery.

2. How to master fractions: What the research shows

As we have seen, simply being able to shade a circular model for a fraction like 3/4 or correctly use terms like numerator and denominator does not imply a student understands a fraction **as a number**. In fact, repeated practice at this rudimentary level may even reinforce thinking about them solely in terms of the numerator and denominator as separate entities: "First I divide the pie into 4 slices; then, I take 3 slices." This explicitly two-step conceptualization can become a barrier to **envisioning the quantity three-fourths as a singular value** – a value that lies halfway between one-half and one on a number line.

A surge in research in the years following the National Math Advisory Panel's 2008 report has led to the identification of more effective instructional practices for fractions. To fully integrate this new number type into an expanded concept of the number system, students need to:

- engage in a progressive series of partitive and iterative activities with models of fractional lengths, including extensive work constructing fractions greater than 1;
- learn to place fractions with a broad range of denominators on a number line, understanding how the number system has now expanded beyond whole numbers to become infinitely dense (an infinite number of values existing between any two points on the number line);
- show proficiency with equivalence and magnitude comparisons between fractions, whole numbers, and mixed numbers, and justify their reasoning using models or representations;
- estimate fractions' positions on number lines, to demonstrate a fully developed number sense with fractions and ability to think of them as numbers in their own right;
- be able to explain rules for operations with fractions using models or representations.

It would seem a tall order to expect students to achieve this level of mastery in today's classrooms, especially given the difficulty historically experienced by so many – but the Department of Education's Institute of Educational Sciences recently examined the published results of a wide range of fractions interventions and evaluated the strength of evidence around various approaches. Interventions backed by strong[1] evidence shared common characteristics – they anchored systematic instruction around a measurement conceptualization of fractions, with number lines as the central representation, in place of the part-whole interpretation that has traditionally been the dominant approach (Fuchs et al., 2021).

2.1 Part-whole and measurement interpretations of a fraction

In a part-whole (parts of a whole) interpretation of a fraction, the whole is typically depicted as a single shape, which is first partitioned into equal-sized parts or pieces, and then some number of those pieces are 'taken out' or shaded to represent a fraction. Circles are a common choice, often in the context of a pizza or pie, as these kinds of foods benefit from an assumed familiarity with slicing into roughly same-sized pieces to get 'equal shares.' Circles are also relatively straightforward to roughly draw and partition into sections accurate enough to communicate the most basic essence of a fraction, at least for small even denominators like 2 and 4. In many classrooms, **shaded area models may mark both the beginning and the end of students' brief conceptual work with representations of fractions** – with numerical notation exclusively employed from then on.

One major limitation of a part-whole foundation is its emphasis on dividing (partitioning) a pre-designated whole into equal-sized parts, corresponding to the denominator, and then shading or otherwise identifying some of these parts to represent the numerator. This often ends up being the sole measure of students' conceptual understanding of fractions – meaning that if they can draw a model of a simple proper fraction, or interpret the fraction shown by such a model, they are deemed ready to progress to numerical representations. Many teachers might be content if their students could reliably do this without occasionally confusing the roles of the numerator and denominator with one another.

As shown earlier through the **Jacob scenario**, difficulties arise when students subsequently try to apply this limited part-whole interpretation to make sense of more advanced concepts. This is because 'parts of a whole (shape)' is only one way that students ultimately need to conceptualize fractions. As first described by Kieran (1976) and elaborated by others such as Behr et al. (1983), there are five key interpretations; **in addition to representing equal-sized parts of a whole, students must also ultimately conceptualize fractions as measurements of length, as ratios, as quotients, and as operators.**

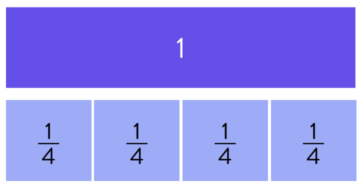
In 2007, Lamon reported on a four-year teaching experiment with five classes of students, each focused on a different one of these interpretations. She concluded the measurement interpretation was the most effective starting point for developing conceptual understanding (Anderson & Boyce, 2015). Subsequent research in the following years amassed considerable evidence around this measurement-based approach as an optimal foundation for learning.

1. strong= **consistent** evidence that these practices **substantially improve** outcomes in **diverse** student populations (Fuchs et al., 2021)

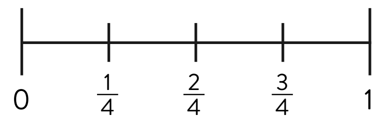
What is a measurement interpretation of a fraction?

> *The interpretation of rational number [fraction] as a measure pushes us beyond our interpretation of a fraction as a part of a whole to the broader idea of a fraction as a quantity compared with a whole. Barnett-Clarke et al. (2010, p. 23), as cited in Neagoy, M. (2017)*

Measurement models are based on an interpretation of a fraction's magnitude as a length or measure. Common representations include:

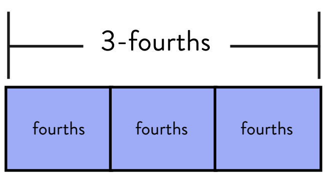


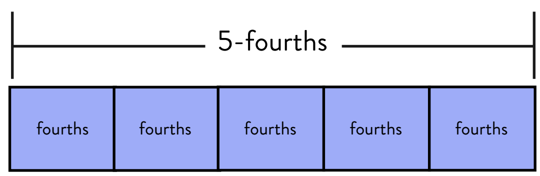
Blocks placed end-to-end



Distance from 0 on a number line

These representations allow for a smooth elaboration to fractional values greater than 1. For example, constructing the fraction 5/4 is understood simply as copying a unit fraction 1/4 block 5 times, the same way 3/4 can be constructed by copying it 3 times:





Copying a fraction in this manner is referred to as iteration.

Key to mastery: extensive partitioning and iteration work with length models

The first step in understanding fractions as numbers requires systematic progression through a range of partitive and iterative activities with models (Wilkins & Norton, 2018), including:

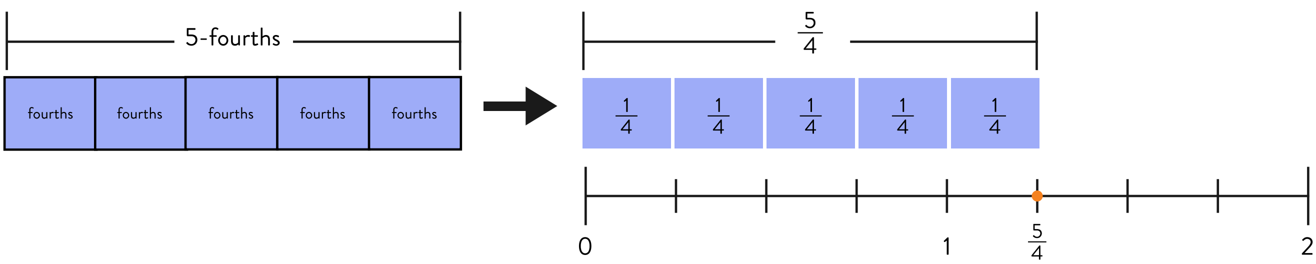
- Partitioning a designated whole into n parts, each representing the unit fraction 1/n;
- Iterating a unit fraction 1/n n times to form n/n, (re)making the whole (n/n = 1);
- Conceptually combining these two acts to make any fraction m/n (including fractions where m > n), by first partitioning the whole to make 1/n and then iterating it m times to make m/n.

Research shows that both circular models and length models can be used for the most simple partitive activities, but **length models are much easier for iteration, or for combining partitioning and iteration**, especially in the context of fractions greater than one (Boyce & Moss, 2022). This may explain why classroom instruction emphasizing the shading of circular models has typically neglected iteration. The **low efficacy of part-whole focused instruction can therefore be linked to both the kinds of models emphasized as well as the limited range of activities done** with these models before students progress to numerical representations.

Experimental studies over the past two decades show measurement-based approaches – where emphasis is placed on partitioning and iterating magnitudes expressed as lengths – consistently lead to superior outcomes. In a recent meta-analysis of over 40 rational number interventions across thousands of students with mathematical difficulties, (Rojo et al. 2023, p. 234) identified a common principle underlying those with positive impact:

> *Initial instruction should focus on developing students' understanding of magnitude through the use of systematic instruction and concrete and semi-concrete length models.*

In addition to being readily usable for concept-building tasks involving partitioning and iteration, another key advantage of length models is that they can be easily related to number lines:



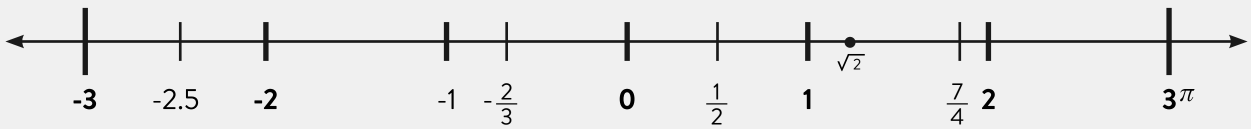
This enables students to make a seamless transition to a more abstract but uniquely powerful representation required for advanced work with fractions.

Does this mean shaded area models should be completely abandoned in fractions instruction?

No. Shaded area models are still typically covered in effective fractions interventions, but as a supplement after primary work employing length representations.

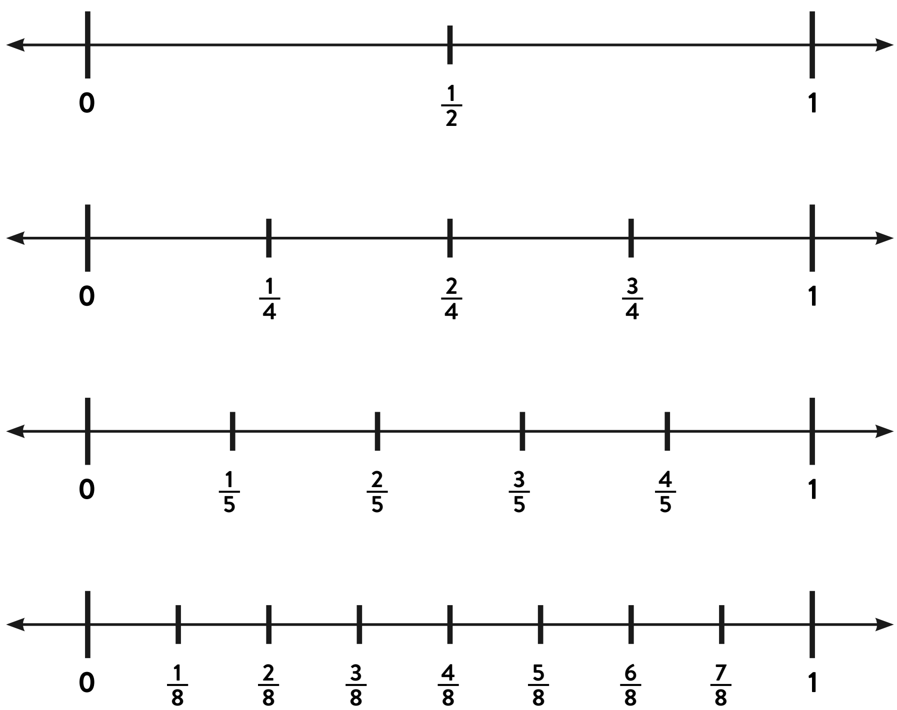
> *Until recently, a part-whole (pie) interpretation of fractions has been the focus within U.S. mathematics curricula (Siegler, Fuchs, Jordan, Gersten, & Ochsendorf, 2015), especially for students with special needs (Gersten, Schumacher, & Jordan, 2017). These curricula may use a number line model, but sparingly. Instead, **we center our instruction on a number line model and use part-whole (pie) models sparingly.** (Dyson et al., 2020, p. 245, emphasis mine)*

2.2 The number line as a central representational tool



The number line is a unique mathematical representation that can concurrently represent all real numbers, including whole numbers and rational numbers, positive and negative numbers, and other sets of numbers (Fuchs et al., 2021, p. 29).

One property that unites whole and rational numbers is that they each have a size (magnitude) determining their place on the number line. This property gives students a way to put any number, including fractions, on a common footing, and to reason about their relative sizes:



Number line examples from Fuchs et al., (2021)

Key to mastery: magnitude comparisons of fractions

Understanding magnitude is a critical aspect of a student's overall understanding of fractions (Siegler & Pyke, 2013). In a study following 475 students from grades 4 through 6, Resnick et al. (2016) demonstrated that **competency with fraction magnitude comparisons was highly predictive of both general fraction knowledge and overall state math test performance** at the end of grade 6. This was true even after statistically controlling for other mathematical abilities, general cognitive characteristics, and demographic variables – leading researchers to conclude that "fraction magnitude understanding is central to mathematical development" (Resnick et al, p. 746).

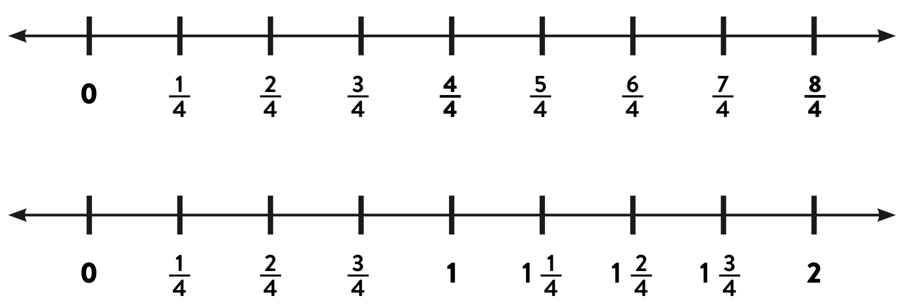
Key to mastery: equivalence, including work with fractions greater than 1 and mixed numbers

Within the broader topic of magnitude comparison lies fraction *equivalence*, a key conceptual milestone in itself. Although seemingly simple, the concept can be difficult for elementary school students to fully acquire and integrate as it involves multiple challenging aspects.

- The equals symbol (=) now signifies equivalence (1/2 = 2/4) rather than solely the outcome of arithmetic (2 + 3 = 5). Research has shown that this new interpretation can itself add sufficient complexity that performance suddenly drops on problems students could do in previous grades (McNeil, 2007).
- A given magnitude is no longer expressed by a single number. (With whole numbers, 2 apples is always just 2 apples, but now an infinite set of numbers are equivalent, e.g., 2/3 = 4/6 = 6/9 = ...).
- The introduction of mixed numbers, which present another new form for expressing a magnitude.

Fortunately, numerous studies over the past two decades have shown that instruction anchored on number lines leads to superior fraction magnitude knowledge among students, especially relative to shaded area models (as cited in Tian et al. (2021): Dyson et al., 2018; Fuchs et al., 2017, 2013, 2014; Gunderson et al., 2019; Hamdan & Gunderson, 2017; Saxe et al., 2007).

For example, Fuchs et al. (2021, p.32) shows how number lines can help students relate equivalent values:



Evidence of number lines' effectiveness in methodologically rigorous studies continues to grow, and the Department of Education's Institute of Education Sciences has correspondingly strengthened its recommendations for their use from moderate (Siegler et al., 2010) to strong (Fuchs et al., 2021).

Estimation using number lines

A particularly powerful way that students can use number lines to develop and demonstrate a deep understanding of numerical magnitude is through estimation of a value's position on it.

Number line estimation (NLE) was initially used to study children's development of a mental representation of whole numbers. A typical NLE experiment uses a number line with just its end points labeled, and participants indicate where they think various values would appear on it:

Where would 16 be on this number line?

Extensive research (e.g., Booth & Siegler, 2006; Cohen & Sarnecka, 2014; Geary et al., 2007; Opfer, Thompson & DeVries, 2007; Siegler & Booth, 2004; Siegler & Opfer, 2003) has shown how most students become increasingly accurate with this task over the early elementary grades. Their development of a mental representation of the number line for whole numbers reflects their maturing number sense and greater capability for proportional reasoning:

> *The more accurate (i.e., linearly spaced) children's mental number line is (as assessed directly by the number line estimation task), the better their performance in other numerical and arithmetic tasks (Link, Hans-Christoph Nuerk, & Moeller, 2014, p. 1599).*

Deceptively simple in appearance, estimation requires multifaceted competencies. Based on their research in NLE across grade levels, Moore & Ashcraft (2015) emphatically stated:

> *...we have come to believe that the number line estimation task is the strongest independent predictor of general math ability available.*

Number sense with whole numbers affects subsequent learning of fractions

In examining growth in fraction knowledge over grades 3-6 with over 500 students, Ye et al. (2016) found that poor NLE performance with whole numbers was a significant predictor for subsequently being in a low-growth trajectory for learning fractions.

Fortunately, research in whole number learning has shown that **NLE activities can be used to improve outcomes, not just predict them** (e.g., Booth & Siegler, 2008; Thompson & Opfer, 2016). In one experimental study, an intervention involving daily practice of NLE for 5 weeks showed improved performance for grade 3 students both with and without dyscalculia, and also transferred to higher rates of solving arithmetical problems correctly (Kucian et al., 2011). Post-intervention analysis via functional magnetic resonance imaging (fMRI) indicated that **NLE practice appeared to lead to reduced cognitive demand** on quantity processing, executive functions, and working memory, and even possible partial remediation of deficient brain activation in participating students with developmental dyscalculia.

Key to mastery: estimation of fractions on number lines

Research shows the value of NLE practice carries over from whole to rational numbers:

Where would $\frac{2}{3}$ be on this number line?

Example of ***fractional value*** *number line estimation* (FNLE)

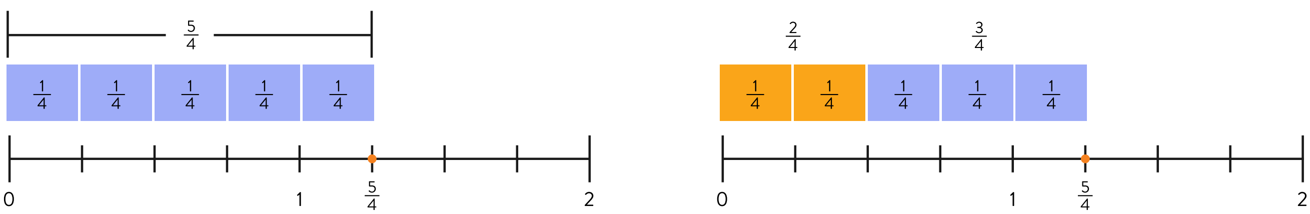
As with NLE, large-scale multiyear studies have examined how proficiency with **fractional value NLE** – also known as **FNLE** – can be a valuable assessment instrument. FNLE is correlated with overall competency with fractions, as well as predictive of future general math achievement:

- In examining at-risk grade 4 students' performance on fraction ordering, Malone & Fuchs (2017) found that FNLE proficiency significantly predicted students' probability of avoiding errors related to whole number bias, whereas performance on questions focused on part-whole understanding did not.
- Hansen et al. (2017) followed over 500 students through grades 4-6, relating performance on fraction magnitude problems, including FNLE tasks, to comprehensive math test scores. The researchers found a bidirectional relation between general mathematics achievement and the fraction magnitude tasks under study, and **FNLE proficiency was the strongest predictor of subsequent general math achievement**.

As with whole numbers, FNLE has also been shown to be a valuable tool for improving performance with fractions. Fazio, Kennedy, & Siegler (2016) demonstrated that practice with a computer-based estimation activity resulted in accuracy gains that then transferred to increased proficiency with magnitude comparisons involving numerical fractions, even improving the ability to recall a fraction previously seen (which can reduce working memory load when doing fraction arithmetic). However, **obtaining these outcomes required students to receive immediate feedback on each estimate** they made during the activity.

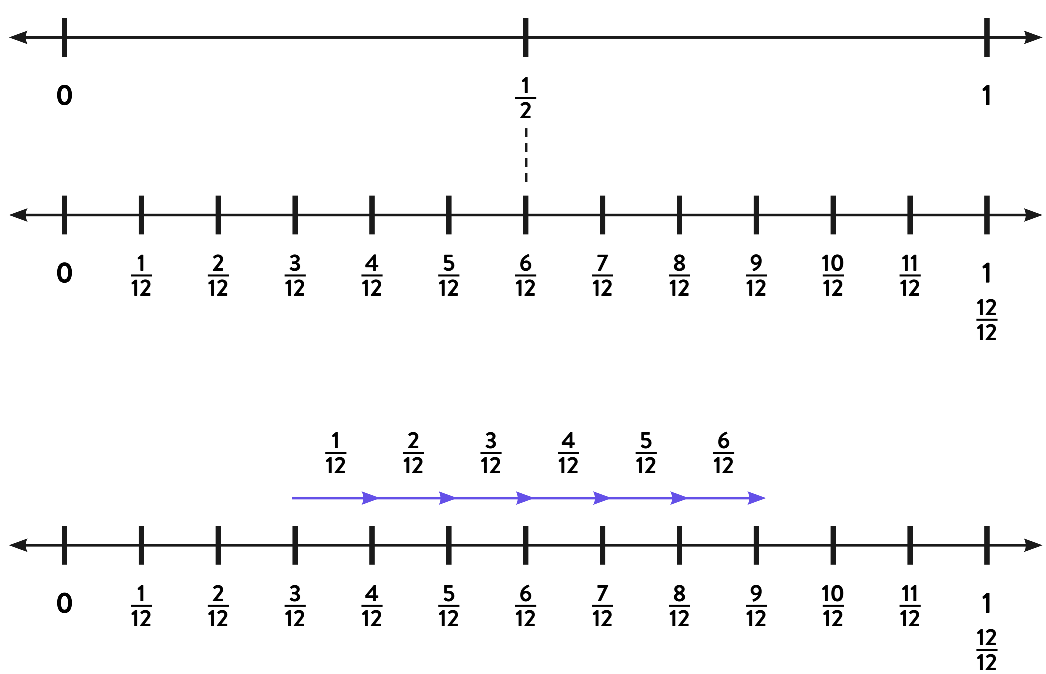
2.3 Making sense of arithmetic with fractions

As the focus shifts to procedural skills in later grades, the measurement-based approach for developing a conceptual understanding of fractions continues to pay significant dividends. Thanks to the extensive iteration activities done with block models at the start of a measurement-based fractions curriculum, addition and subtraction with like denominators now seems intuitive and natural. It is nothing new, just the same work students initially engaged in while iterating blocks, now expressed in numerical form:



Similarly, with a more robust understanding of equivalence, the rationale for addition and subtraction with unlike denominators also follows an understandable underlying logic. As the IES Practice Guide (Fuchs et al., 2021, p. 37) illustrates, number lines can be used to model the procedure for adding an expression like 3/12 + 1/2 by:

- first showing that 6/12 is equivalent to 1/2,
- then demonstrating how to add 6/12 to 3/12 to find the sum, 9/12



Key to mastery: relating fraction arithmetic to measurement models and number lines

Even when computational procedures themselves are not part of an intervention, research has shown that measurement-based approaches have a positive impact on proficiency with fraction arithmetic.

An early example is Fuchs et al. (2013), reporting the results of research involving several hundred grade 4 students. The study compared the efficacy of a 36-session measurement-based intervention against control groups learning from a high-quality 'business-as-usual' fractions curriculum. As in other recent studies of this kind, the measurement-based intervention led to superior improvement across outcome measures. An analysis of at-risk student subgroups revealed that the performance gap between them and more typically achieving students was significantly narrowed or eliminated. Particularly notable was that **performance on numerical procedures was dramatically improved, even though the intervention itself only focused on conceptual understanding and fraction number sense**. Here again, the researchers found that students' improvement in fractional value number line estimation **(FNLE) was a particularly strong factor** in the intervention's overall effectiveness.

Measurement models continue to support students as they progress onwards to learn about *multiplication* and *division* involving fractions. Multiplication of a fraction by a whole number can now be readily understood as the familiar process of iterating a unit or non-unit fraction, and division of a whole number by a fraction as partitioning (Lesner et al., 2023). Through this connection to students' fundamental understanding of fractions as numbers, arithmetic with fractions can first be represented and discussed using block models, and then progressed to number lines as a final representational step. From there, students should be more ready to engage with abstract examples in purely numeric form.

The number line thus becomes the chief and unifying representation across K-8, as recommended by the Institute of Education Sciences (Fuchs et al., 2021). Having a single visual representation as a common reference point for the wide range of concepts and skills covered during the elementary grades helps students more easily navigate the introduction of rational numbers and integrate them into an evolved concept of the number system. As Jayanthi et al. summarize (2021, p. 98, emphasis mine):

> ***Most curricula include a wide array of visual representations for teaching fractions****, which can be* ***overwhelming to teachers*** *in choosing how to represent fractions concepts during instruction...*

The number line is of particular importance for teaching fractions concepts — a finding that is supported not only by this study but also by a long history of intervention research incorporating the number line to teach fractions, especially fraction magnitude (e.g., Barbieri et al., 2020; Fuchs et al., 2016; Malone et al., 2019). Students are likely to benefit from the **consistent use of the number line to teach not just foundational fractions concepts, but also the reasoning underlying computational procedures for all four operations.**

2.4 Research into practice: implementation challenges

Over the past two decades, groundbreaking experimental research has pioneered a number of innovative pedagogical approaches that lead to genuine mastery of fractions. However, while they have been clearly shown to work well in controlled settings, significant challenges remain in implementing them in the broader setting of a classroom.

Of course, the need to deal with an assortment of practical issues in applying research-based pedagogical methods is not a new phenomenon, nor one solely encountered in the context of mathematics education. Classroom realities have always played an understandable role in dictating the ways in which teachers traditionally cover fractions. For example, circular area models are relatively quick and easy to draw for both teachers and students, at least for even-numbered values with small denominators. This convenience has presumably been a primary reason why they have been a mainstay of students' initial exposure to fractions – making demonstrations of the meaning of *one-half* and *three-fourths* using 'pies' or 'pizzas' a familiar rite of passage in elementary schools across the U.S.

We now know that effective instruction for developing number sense with fractions requires students to engage in a progressive series of both *partitive* and *iterative* activities. Research has shown that **as these modeling tasks become more complex, they become increasingly easier to comprehend when done with length models** compared to circular area models (e.g., Boyce & Moss, 2022). Regardless of model type, however, students still need sufficient direct guidance to successfully complete conceptually-focused activities. For example, the Institute of Education Sciences mathematics intervention guide stresses such practical considerations as reminding students to focus on the correct 'ends' of a block model (Fuchs et al., 2021) – i.e., the ones that correspond to its length rather than its height. As students progress from physical block models to paper-based work in the classroom, **length models can be challenging to draw with sufficient accuracy to support the development of proportional reasoning**.

A major transition in effective fractions instruction using a measurement-based approach occurs during the progression from length models to the more abstract representation of a number line. As with length models, studies have shown that **number lines must be drawn with a certain accuracy to be effective**, even when working with whole numbers rather than fractional ones (e.g., Booth & Siegler, 2008). In addition, to internalize the infinitely dense nature of rational numbers, students must work with a wide range of denominators on number lines – including odd-numbered denominators, which are sometimes avoided due to the greater difficulty in drawing them accurately on paper (Fuchs et al., 2021).

Timely feedback is also essential, particularly during activities involving estimation. Studies analyzing learning gains have found that without immediate feedback on the accuracy of their responses, students' performance did not improve – a finding consistent across research on number line estimation activities with younger students involving whole numbers (e.g., Thompson & Opfer, 2016) as well as older ones working with fractions (e.g., Fazio, Kennedy, & Siegler, 2016). This raises another practical challenge, as delivering immediate feedback on an individual basis can be next to impossible when dealing with a class full of students simultaneously doing a number line activity. This has likely been a factor in why estimation activities have been relatively rarely pursued in classroom instruction, despite their proven benefits in both learning and assessment.

On a more general level, proficiency with fractions involves the acquisition of a very wide range of concepts and skills. As a consequence, even individually powerful learning activities like estimation need to be integrated within a comprehensive, systematic course of study for full effect; as noted by Nuraydin et al. (2022), "gamified NLE interventions can complement but not replace more direct instruction on fraction concepts and fraction arithmetic".

Given the historical dominance of part-whole interpretations of fractions and circular area models, **many of today's teachers may not have personal experience learning from measurement-based interpretations or doing extensive number line work** from their own elementary school years as students. Some of the representations used in new research-grounded instructional methods may be unfamiliar, as will activities like estimation. Adopting a measurement-based approach in the classroom may initially require significant support around these novel aspects.

The same may be true with respect to transitioning **to a mastery-focused mindset with fractions instruction**. Existing teaching practices may have placed limited emphasis on developing genuine fraction number sense, instead progressing almost immediately to drill-and-practice with numerical procedures and rote memorization of decontextualized 'tricks' such as 'invert and multiply.' Changing from business as usual to a new approach may require a concerted and coordinated effort at an organizational level.

A final obstacle is the **time commitment** that a pursuit of fractions mastery may be perceived to require, particularly with respect to at-risk students. Although the recommendations of the NMAP, NCTM and Common Core have all called for a streamlining of the mathematics curriculum to make adequate room for all students to attain sufficient competency with fractions, there are still many demands placed on classroom time in elementary schools, including those from subjects other than mathematics. In this regard, **any proposed solution intended for general classroom use must achieve both efficacy and efficiency**.

3 From research to classroom reality with Frax

Strong evidence shows that measurement-based interventions with small groups of elementary students can achieve dramatic results. Today's challenge is to integrate these research-validated methods into a single comprehensive system that is practical for widespread classroom use.

ExploreLearning developed Frax to address this need, focusing on the **key fundamentals of fractions that are best introduced and mastered with technology support**. Number lines are a prime example of how an interactive, adaptive learning environment like Frax can profoundly increase the accessibility of a critical tool for mathematical exploration.

3.1 Number lines – making this powerful abstract representation accessible to all

In their journey to fractions mastery, students must ultimately become proficient with the number line as a primary representation (Fuchs et al., 2021). Its abstract nature is challenging for many, however, particularly those with math difficulties (Rodrigues et al., 2016). Therefore, a key aspect of an effective instructional methodology for fractions is how it scaffolds the development of competency with this representation. Frax provides this scaffolding in a careful and natural way:

- The introduction of fractions begins with simple length (block) models, which are a much more concrete and accessible starting point than number lines. Students do extensive work constructing models through partitioning and iteration, progressing from unit to non-unit fractions, and to fractions greater than one. The numerical form of a fraction is gradually introduced in relation to these models.
- Students then begin making magnitude comparisons with models, as well as producing models that satisfy given magnitude characteristics. Important learning milestones here include fluid reasoning regarding numerators' and denominators' respective roles, and how they work together to create a magnitude.
- A gradual progression to number lines then begins, which is made more intuitive by guiding students to relate intervals drawn between whole numbers back to the now-familiar block models – giving them the means to conceptualize and interpret the fractional tick marks on a number line in a less abstract way.
- In this manner, the number line becomes accessible for all students to benefit from as they progress into more challenging magnitude-related activities, such as those involving equivalency.

Within Frax, rich interaction with accurate length models and number lines engages students in the full range of partitioning and iterating activities necessary to develop a robust number sense with fractions and integrate it into an expanded concept of the number system. Problems are delivered in individualized sequences with immediate, tailored feedback and adaptive support that continually deepens understanding and accelerates progress.

Another example of the importance of technology-supported learning with number lines is estimation. As previously summarized in this white paper, extensive research has shown how estimation is not only a valuable assessment of students' number sense, but also forms the basis for highly effective activities to develop it. Harnessing the power of estimation in the classroom can be challenging, however, as it places an even greater premium on immediate feedback to each response's accuracy, as well as on individualized progressions that gradually but efficiently increase the challenge as a student develops competency.

Technology brings key representations into reach for teaching, unlocking their potential as tools in classroom discussion

A key benefit of using technology to provide a solid initial grounding in the use of block models and number lines is that their use is subsequently **no longer bound to computers.** Once the fundamentals of these representations are understood and mastered, students and teachers can begin to draw much more approximate models in the classroom in a draft, freehand manner to discuss fraction concepts and model procedures on the fly, including diagramming word problems and modeling the logic underlying procedures for fraction arithmetic.

Frax directly supports this important benefit through activities that scaffold the transition from a computer-based learning environment to pencil and paper, avoiding a common stumbling block in systems that do not actively cover this "last mile" in the learning process.

This is particularly important for schools that have adopted research-based curricula emphasizing number lines in classroom instruction and student activities. When teachers encounter challenges in developing students' basic competency with number lines on paper, they may understandably be tempted to reach for what's familiar, falling back on business-as-usual practices such as traditional shaded circular models.

3.2 Efficacy + efficiency = impact

The Frax system packages the core fundamentals of fractions mastery into two successive levels, *Foundations I* and *Foundations II*:

- Foundations I is designed as a 'zero-entry' program so that students with no previous knowledge of fractions can begin using the program immediately.
- Each level consists of 25-30 consecutive 30-minute Missions. A rate of 3-5 Missions a week is recommended, allowing students to complete either level in approximately 2 months.
- This enables educators to make strategic investments of 10-15 instructional hours per level to ensure all students master the fraction fundamentals that are most effectively learned through technology.

Given the importance of these fundamentals, the best return on instructional time occurs if Frax is prioritized by implementing it prior to a fractions unit, laying a solid foundation for successful math instruction and the grade-level objectives that follow.

3.3 Designed for active learning and sustained engagement

ExploreLearning developed Frax to make powerful new research-validated methods for fractions mastery practical in the classroom – and fun for students.

As with the ExploreLearning Reflex approach to math fact fluency, students using Frax benefit from highly interactive and adaptive educational technology that harnesses the power of:

- Learning by doing – instead of passively watching or listening, students actively and continuously engage with virtual models that develop their conceptual reasoning ability.
- Immediate just-in-time feedback and support, tailored to students' specific responses.
- Individualized progression through challenges of increasing difficulty, driven by each student's needs and demonstrated capabilities.

Like Reflex, Frax delivers proven pedagogy within a highly motivating system:

- Game-based learning – well-designed activities that continually reveal new surprises over time – particularly important for students who struggle to maintain focus with traditional forms of classroom practice.
- Recognition of effort as well as progress – students earn tokens for use in the Frax Store to customize their personal cabin on the Frax starship, the "F.F.S. Sable".
- Engaging storylines with memorable characters – each Frax Mission is anchored around a themed adventure that connects its activities together.

3.4 Classroom results using Frax

Launched in 2021, Frax is already bringing fractions mastery to thousands of schools across the U.S.

Frax teacher survey responses

ExploreLearning's Educator Grant program gives teachers free access to Frax for one school year. Of the 1,465 teachers awarded a grant in the 2021–22 school year:

- **99.7%** responded that they saw **improvement in student learning and engagement** because of Frax;
- **87%** stated Frax was **better than any other program or tool they previously used to teach fractions**.

The most frequently observed improvements were increased understanding of fractions concepts, confidence in math abilities, and enjoyment in learning.

Even larger improvements were seen in the 72 schools reported as having especially low standardized test scores. Teachers there reported the same or statistically larger gains with Frax relative to peers in higher-scoring schools, and noted greater student participation in class and increased self-esteem.

ESSA Tier 2 study results

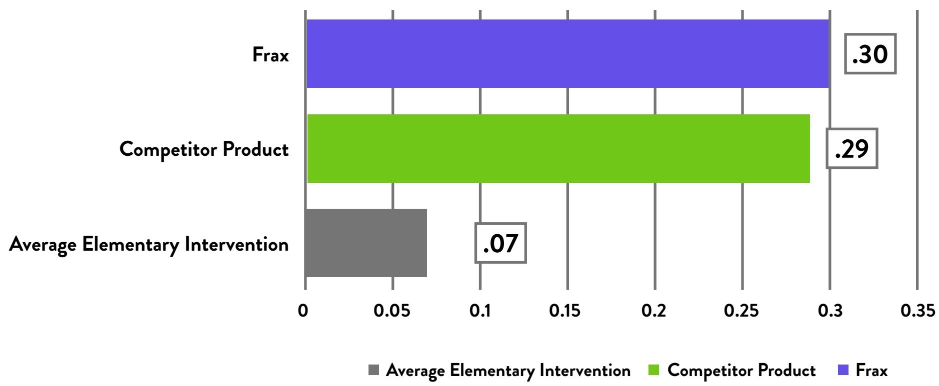
Formal research measuring Frax's impact on student achievement was conducted in a large Florida school district during the 2021–22 school year. The study, certified as ESSA Tier 2, involved 2,530 students in grades 3 and 4 using Frax Foundations I, matched against an equivalent sample of 2,530 students using business-as-usual fractions instruction.

A key finding was that all students who used Frax were significantly more likely to reach grade level proficiency than matched non-users, even those starting with a 2+ grade level deficit:

Placement level at Fall '21	Spring '22 results – Frax users compared to matched non-users:
2+ grade levels below	2x as many Frax users reached or approached Spring '22 grade level
1 grade level below	44% more Frax users reached Spring '22 grade level
On or approaching grade level	28% more Frax users reached Spring '22 grade level

Frax was more than 4x effective than the average elementary math intervention, and was highly efficient relative to a leading adaptive solution for individualized practice (Shah et al., 2023), **achieving superior results in only a quarter of the usage time**:

Hedge's g effect size of product usage and scale score gains



Frax shows that with a relatively modest investment of time—on average 13 hours for most students to complete Sector 1 on foundational understanding—students of all ability levels really can succeed with fractions, creating immediate achievement gains – and even more importantly, removing a major obstacle to future success and opportunity.

For more on this study and additional Frax research results, visit:

www.fraxmath.com/about/research-behind-frax

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